

# NON SYMMETRIC DIRICHLET FORMS ON SEMIFINITE VON NEUMANN ALGEBRAS

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**Abstract.** The theory of non symmetric Dirichlet forms is generalized to the non abelian setting, also establishing the natural correspondences among Dirichlet forms, sub-Markovian semigroups and sub-Markovian resolvents within this context. Examples of non symmetric Dirichlet forms given by derivations on Hilbert algebras are studied.

## Introduction.

The theory of non commutative Dirichlet forms, which originated from the pioneering examples of L. Gross [G] and the general analysis of S. Albeverio and R. Høegh-Krohn [AH] (see also [AHO]), has nowadays drawn a renewed interest between researchers ([DL1], [DL2], [DR], [D3], [Sa], [GL] and [Ci]). There are different reasons which, in our opinion, explain (and justify) the recent activity in this area. On the one side the presence of a feed-back effect due to the increasing ability showed by the commutative theory in handling successfully analytic and probabilistic problems during the last fifteen years ([AR], [AMR], [MR], [D2], and ref. therein). On the other side the great recent development of other new branches of mathematics such as A. Connes’ non commutative geometry ([Co] and ref. therein) and quantum probability ([Pa], [AW] and ref. therein) with which the theory of non commutative Dirichlet forms can naturally fit in. Let us remark that up to now all works on non commutative Dirichlet forms treated the generalization to a non abelian setting of the symmetric classical theory (see [F]).

In this paper, we develop the general theory of non symmetric Dirichlet forms on a semifinite von Neumann algebra  $\mathcal{A}$ . This means that we study sesquilinear forms on the Hilbert space  $L^2(\mathcal{A}, \tau)$ , requiring only the so called “weak sector condition” , which, at the form level, roughly means the antisymmetric part of the form must be controlled by the symmetric one. In this sense our work can be seen as a non commutative extension of the theory of Dirichlet forms as it has been recently presented in [MR], where this condition is assumed from the very beginning. It is worthwhile to notice that these authors are able to produce a large amount of examples of Dirichlet forms (see [MR] Chap. II). Among their examples let us quote the following simple one: consider the form

$$\mathcal{E}(u, v) := \sum_{i,j=1}^n \int a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \quad (0.1)$$

where  $u$  and  $v$  are  $\mathcal{C}^\infty$  functions with compact support in an open set of  $\mathbf{R}^n$ . If the functions  $a_{ij}(x)$  are locally summable on  $U$ , the symmetric part  $[\tilde{a}_{ij}(x)]$  of the matrix-valued function  $[a_{ij}(x)]$  is uniformly bounded from below by a positive constant and the entries of the antisymmetric part  $[\check{a}_{ij}(x)]$  are  $L^\infty$  functions, it can be proven that the form  $\{\mathcal{E}, D(\mathcal{E})\}$  is closable and its closure is a Dirichlet form. As it will be explained in Section 5, a natural generalization of the preceding example is given by the form

$$\mathcal{E}(x, y) := \sum_{i,j=1}^n (d_j x, a_{ij} d_i y) \quad (0.2)$$

where  $a_{ij} := \delta_{ij} + c_{ij}$  and  $[c_{ij}]$  is an antisymmetric matrix whose entries are in the center of  $\mathcal{A}$ . As a consequence of the theory developed in this paper, we are able to prove that, if  $d_i$  are closable derivations and the intersection of their domains is dense, then the form in (0.2) gives rise to a Dirichlet form.

To better illustrate our results, we recall that throughout this paper  $\mathcal{A}$  is a von Neumann algebra with a faithful, normal, semifinite trace  $\tau$ . Forms, semigroups, resolvents etc. are defined on the complex Hilbert space  $L^2(\mathcal{A}, \tau)$ , even though many of their properties and relations require the real Hilbert space  $L^2(\mathcal{A}, \tau)_h$  and its underlying order structure in an essential way.

In Section 1 we collect some preliminary material taken from [MR] on the relationships between coercive closed forms on a Hilbert space and strongly continuous contraction resolvents (resp. semigroups and their generators) satisfying sector condition. The last paragraphs of the section recall the essentials of I.E. Segal’s theory of non commutative  $L^p$  spaces on  $\mathcal{A}$  (see [N], [Se], [St]).

In section 2 we establish the correspondence between Dirichlet forms, sub-Markovian semigroups and sub-Markovian resolvents, thus generalizing the results of S. Albeverio and R. Høegh-Krohn ([AH], see also [DL1]) to the non symmetric case. This is

made adapting the non-symmetric abelian definitions and results in [MR] to the non commutative (semifinite) case.

Section 3 is devoted to the extension of some properties of sub-Markovian semigroups, already studied in [DL1], to the non symmetric context. In particular we prove that sub-Markovian semigroups may be extended to  $L^p$  spaces and study a class of sub-Markovian semigroups on  $L^\infty(\mathcal{A}, \tau)$ , showing a correspondence between such semigroups and those on  $L^2(\mathcal{A}, \tau)$ . Finally we study the consequences of complete positivity for semigroups and Dirichlet forms such as the contraction property for semigroups on  $L^\infty(\mathcal{A}, \tau)$ .

In Section 4 we study derivations on Hilbert algebras and, based on previous results in [Sa] and [DL1], we prove that, for a closed derivation on a Hilbert algebra, the self-adjoint part of its domain is closed under Lipschitz functional calculus and the whole domain is closed under the modulus operation. We also show that the corresponding norm inequalities (see (4.4) and (4.6)) hold, *i.e.* such a derivation is a Dirichlet derivation in the sense of E.B. Davies and J.M. Lindsay [DL1]. Moreover, a non-abelian chain rule holds for the  $\mathcal{C}^1$  functional calculus of a self-adjoint operator. We notice that  $\delta$  need not be a  $*$ -derivation for the previous results to hold. Finally we show how derivations which are not  $*$ -invariant give rise naturally to (non symmetric) Dirichlet forms.

In section 5 we prove a theorem which gives rise to new examples of non commutative Dirichlet forms (and related semigroups). These examples are of the previously mentioned type. They were already studied in [DL1] in the symmetric case: this simply corresponds to requiring the antisymmetric part  $[c_{ij}]$  in (0.2) to vanish.

Lastly let us mention that these results may be useful in the context of open quantum systems and quantum statistical mechanics which, as it is known, represent a natural physical arena where these mathematical theories have found interesting applications (see e.g. references in [AH] and [DL1]).

## Section 1. Preliminaries.

In this section we first collect definitions and facts about strongly continuous semigroups and related objects, referring to [MR] for proofs and further results, and then definitions and facts about  $L^p$  spaces on  $\{\mathcal{A}, \tau\}$ , a von Neumann algebra with a faithful semifinite normal trace, referring to classic works of [Se], [N] and [St] for more detailed analysis and proofs, and to [T] for the general theory of von Neumann algebras.

It is well known that there is a bijective correspondence between strongly continuous contraction resolvents  $\{G_\alpha\}_{\alpha>0}$  on a Banach space  $X$ , strongly continuous contraction semigroups  $\{T_t\}_{t>0}$  on  $X$ , and closed, densely defined linear operators  $\{L, \mathcal{D}(L)\}$  on  $X$ , with the properties that  $(0, \infty) \subset \rho(L)$ , and  $\|\alpha(\alpha - L)^{-1}\| \leq 1$ ,  $\forall \alpha > 0$ . These objects are related by

$$\begin{aligned} G_\alpha &= (\alpha - L)^{-1}, \quad \alpha > 0 \\ G_\alpha x &= \int_0^\infty e^{-\alpha t} T_t x dt, \quad x \in X \\ Lx &= \lim_{t \downarrow 0} \frac{T_t x - x}{t}, \quad x \in \mathcal{D}(L) := \{x \in X : \lim_{t \downarrow 0} \frac{T_t x - x}{t} \text{ exists}\} \\ T_t x &= \lim_{\alpha \rightarrow \infty} T_t^{(\alpha)} x := \lim_{\alpha \rightarrow \infty} e^{-\alpha t} \sum_{n=0}^\infty \frac{(t\alpha)^n}{n!} (\alpha G_\alpha)^n x, \quad x \in X. \end{aligned} \tag{1.1}$$

Recall now the theory of coercive closed forms.

Let  $\mathcal{H}$  be a complex Hilbert space,  $\mathcal{K} \subset \mathcal{H}$  a real vector subspace s.t.  $\mathcal{K} + i\mathcal{K} = \mathcal{H}$  and  $(x, y) \in \mathbf{R}$ ,  $\forall x, y \in \mathcal{K}$ , and denote with  $M_h := M \cap \mathcal{K}$ , the real part of  $M \subset \mathcal{H}$ , and with  $x^* := y - iz$ , the adjoint of  $x = y + iz$ ,  $y, z \in \mathcal{K}$ .

Let  $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \rightarrow \mathbf{C}$ , where  $\mathcal{D}(\mathcal{E})$  is a subspace of  $\mathcal{H}$ , be a real-positive, sesquilinear form on  $\mathcal{H}$ , that is,  $\forall x, y, z \in \mathcal{D}(\mathcal{E})$ ,  $\alpha, \beta \in \mathbf{C}$ ,

$$\begin{aligned} \mathcal{E}(x, \alpha y + \beta z) &= \alpha \mathcal{E}(x, y) + \beta \mathcal{E}(x, z) \\ \mathcal{E}(y, x) &= \overline{\mathcal{E}(x, y)} \\ \mathcal{E}(x^*, y^*) &= \overline{\mathcal{E}(x, y)}, \end{aligned}$$

and

$$\mathcal{E}(x, x) \geq 0, \quad x \in \mathcal{D}(\mathcal{E})_h,$$

and denote by  $\tilde{\mathcal{E}}$  the symmetric part of  $\mathcal{E}$ ,

$$\tilde{\mathcal{E}}(x, y) := \frac{1}{2}[\mathcal{E}(x, y) + \overline{\mathcal{E}(y, x)}], \quad x, y \in \mathcal{D}(\mathcal{E}),$$

and by  $\check{\mathcal{E}}$  the antisymmetric part of  $\mathcal{E}$ ,

$$\check{\mathcal{E}}(x, y) := \frac{1}{2}[\mathcal{E}(x, y) - \overline{\mathcal{E}(y, x)}], \quad x, y \in \mathcal{D}(\mathcal{E}).$$

Finally, denote by  $\mathcal{E}_\alpha$ ,  $\alpha \geq 0$ , the form  $\mathcal{E}_\alpha(x, y) := \mathcal{E}(x, y) + \alpha(x, y)$ ,  $\forall x, y \in \mathcal{D}(\mathcal{E})$ .

**1.1 Definition.**  $\{\mathcal{E}, \mathcal{D}(\mathcal{E})\}$  is said to satisfy the weak sector condition if  $\exists K > 0$  s.t.  $|\mathcal{E}_1(x, y)| \leq K\mathcal{E}_1(x, x)^{1/2}\mathcal{E}_1(y, y)^{1/2}$ ,  $x, y \in \mathcal{D}(\mathcal{E})_h$ .

Notice that the above definition is equivalent to:  $\exists K' > 0$  s.t.  $|\check{\mathcal{E}}_1(x, y)| \leq K\mathcal{E}_1(x, x)^{1/2}\mathcal{E}_1(y, y)^{1/2}$ ,  $x, y \in \mathcal{D}(\mathcal{E})_h$ .

**1.2 Definition.**  $\{\mathcal{E}, \mathcal{D}(\mathcal{E})\}$  is said a coercive closed form on  $\mathcal{H}$  if

- (i)  $\mathcal{D}(\mathcal{E})$  is dense in  $\mathcal{H}$
- (ii)  $\{\check{\mathcal{E}}, \mathcal{D}(\mathcal{E})\}$  is closed [i.e.  $\{\mathcal{D}(\mathcal{E}), \check{\mathcal{E}}_1\}$  is a Hilbert space]
- (iii)  $\{\mathcal{E}, \mathcal{D}(\mathcal{E})\}$  is real-positive and satisfies the weak sector condition.

**1.3 Definition.** A positive linear operator  $\{A, \mathcal{D}(A)\}$  on  $\mathcal{H}$  is said to satisfy the sector condition if  $\exists K > 0$  s.t.  $|(x, Ay)| \leq K(x, Ax)^{1/2}(y, Ay)^{1/2}$ ,  $x, y \in \mathcal{D}(A)_h$ .

**1.4 Theorem.** There is a bijective correspondence between coercive closed forms  $\{\mathcal{E}, \mathcal{D}(\mathcal{E})\}$  and strongly continuous contraction resolvents  $\{G_\alpha\}_{\alpha>0}$  s.t.  $G_\alpha$  satisfies the sector condition for some (hence for all)  $\alpha > 0$ .

These objects are related by

$$\mathcal{E}_\alpha(x, G_\alpha y) = (x, y), \quad x \in \mathcal{D}(\mathcal{E}), \quad y \in \mathcal{H},$$

and, if  $L$  is the generator of  $\{G_\alpha\}_{\alpha>0}$ ,

$$\mathcal{E}(x, y) = (x, -Ly), \quad x \in \mathcal{D}(\mathcal{E}), \quad y \in \mathcal{D}(L),$$

where  $\mathcal{D}(\mathcal{E})$  is the completion of  $\mathcal{D}(L)$  w.r.t.  $\check{\mathcal{E}}_1^{1/2}$ .

**1.5 Proposition.** Let  $\{\mathcal{E}, \mathcal{D}(\mathcal{E})\}$  be a coercive closed form on  $\mathcal{H}$ , and  $\{G_\alpha\}_{\alpha>0}$ , the associated resolvent. Then, setting  $\mathcal{E}^{(\beta)}(x, y) := \beta(x, y - \beta G_\beta y)$ ,  $x, y \in \mathcal{H}$ , we get

- (i)  $|\mathcal{E}_1^{(\beta)}(x, y)| \leq (K + 1)\mathcal{E}_1(x, x)^{1/2}\mathcal{E}_1^{(\beta)}(y, y)^{1/2}$ ,  $x \in \mathcal{D}(\mathcal{E})$ ,  $y \in \mathcal{H}$
- (ii) Let  $x \in \mathcal{H}$ . Then  $x \in \mathcal{D}(\mathcal{E}) \iff \sup_{\beta>0} \mathcal{E}^{(\beta)}(x, x) < \infty$
- (iii)  $\lim_{\beta \rightarrow \infty} \mathcal{E}^{(\beta)}(x, y) = \mathcal{E}(x, y)$ ,  $x, y \in \mathcal{D}(\mathcal{E})$ .

Let  $\mathcal{A}$  be a semifinite von Neumann algebra with a faithful normal semifinite trace  $\tau$ , and let  $(\pi, \mathcal{H}, \Lambda)$  its GNS representation. From now on  $\mathcal{A}$  is identified with its representation  $\pi(\mathcal{A})$ .

Let  $\Xi$  be the collection of the closed, densely defined operators on  $\mathcal{H}$  affiliated with  $\mathcal{A}$ . Then, by spectral theorem,  $\forall x \in \Xi_h$ ,

$$x = \int_{-\infty}^{+\infty} \lambda de_x(\lambda),$$

where  $e_x(E) \in \mathcal{A}$  for any Borel subset  $E$  of  $\mathbf{R}$ , therefore

$$\nu_x(E) := \tau(e_x(E))$$

is a Borel measure on  $\mathbf{R}$  and

$$\tau(x) := \int_0^{+\infty} \lambda d\nu_x(\lambda), \quad x \in \Xi_+$$

is a faithful extension of  $\tau$  to  $\Xi_+$ .

Now, let us define,  $\forall x \in \Xi$ , and for  $p \in [1, \infty)$ ,  $\|x\|_p := \tau(|x|^p)$ .  
Let, for each  $p \in [1, +\infty)$ ,

$$L^p(\mathcal{A}, \tau) := \{x \in \Xi : \|x\|_p < +\infty\}$$

and,

$$\tilde{\mathcal{A}} := \{x \in \Xi : \nu_{|x|}((\lambda, +\infty)) < +\infty \text{ for some } \lambda > 0\}.$$

Finally set  $L^\infty(\mathcal{A}, \tau) := \mathcal{A}$ . It turns out that  $\tilde{\mathcal{A}}$ , equipped with strong sense operations [Se] and with the topology of convergence in measure ([St], [N]), becomes a topological  $*$ -algebra, called the algebra of  $\tau$ -measurable operators. Moreover  $\{L^p(\mathcal{A}, \tau), \|\cdot\|_p\}$  is a Banach subspace of  $\tilde{\mathcal{A}}$ , is linearly spanned by its positive elements, and its norm satisfies  $\|x\|_p = \|x^*\|_p$ , for all  $x \in L^p(\mathcal{A}, \tau)$ . Finally, observe that  $L^2(\mathcal{A}, \tau)$  is a Hilbert space, with the scalar product given by  $(x, y) := \tau(x^*y)$ ,  $x, y \in L^2(\mathcal{A}, \tau)$ .

The basic properties of the  $L^p$  spaces are:

**1.6 Proposition.** (i) *The trace  $\tau$  is extended to  $L^1(\mathcal{A}, \tau)$  by linearity, and*

$$\begin{aligned} |\tau(x)| &\leq \tau(|x|) = \|x\|_1 \\ \tau(x^*) &= \overline{\tau(x)} \end{aligned} \quad x \in L^1(\mathcal{A}, \tau)$$

(ii) *For each  $x \in \Xi_h$  and for each Borel measurable function  $\varphi : \mathbf{R} \rightarrow \mathbf{C}$  one has*

$$\|\varphi(x)\|_p = \|\varphi\|_{L^p(\mathbf{R}, \nu_x)}.$$

*In particular, if  $\varphi \geq 0$  or  $\varphi \in L^1(\mathbf{R}, \nu_x)$  we get*

$$\tau(\varphi(x)) = \int \varphi d\nu_x$$

- (iii)  $x, y \in L^p(\mathcal{A}, \tau)_+, x \leq y \Rightarrow \|x\|_p \leq \|y\|_p$ .
- (iv)  $x, y \in \tilde{\mathcal{A}}, p, q, r \in [1, +\infty], \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \Rightarrow \|xy\|_r \leq \|x\|_p \|y\|_q$
- (v) Let  $p \in [1, +\infty)$ . Then for each  $\psi \in L^p(\mathcal{A}, \tau)^*$  there exists a unique element  $x_\psi \in L^{p'}(\mathcal{A}, \tau)$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ , such that  $\langle \psi, y \rangle = \tau(x_\psi y)$  and  $\psi \in L^p(\mathcal{A}, \tau)^* \rightarrow x_\psi \in L^{p'}(\mathcal{A}, \tau)$  is a Banach space isomorphism.
- (vi) Let  $p \in [1, +\infty], x \in L^p(\mathcal{A}, \tau), y \in L^{p'}(\mathcal{A}, \tau)$ ; then

$$\begin{aligned}\tau(xy) &= \tau(yx) \\ \tau(xy) &\geq 0, \quad x, y \geq 0\end{aligned}$$

- (vii) The weak\* topology on  $L^\infty(\mathcal{A}, \tau)$  given by the duality  $\langle L^1(\mathcal{A}, \tau), L^\infty(\mathcal{A}, \tau) \rangle$  coincides with the  $\sigma$ -weak topology.

Let us denote with  $Proj(\mathcal{A})$  the set of self-adjoint idempotents of  $\mathcal{A}$  and with  $\mathcal{S}$  the following set

$$\mathcal{S} := \{x = \sum_{i=1}^n \lambda_i e_i \in \mathcal{A} : n \in \mathbf{N}, \lambda_i \in \mathbf{C}, e_i \in Proj(\mathcal{A}) \cap L^1(\mathcal{A}, \tau)\}.$$

### 1.7 Proposition.

- (i)  $\mathcal{S}_+$  is dense in  $L_+^p$  for  $p \in [1, \infty)$  and weak\* dense in  $L_+^\infty$
- (ii)  $x \in L^p, p \in [1, \infty] \Rightarrow x e_{|x|}((\frac{1}{n}, n)) \rightarrow x$  in  $L^p$
- (iii) Let  $x \in \tilde{\mathcal{A}}$ ; if  $ax \in L^1$  and  $\tau(ax) = 0, \forall a \in L^1(\mathcal{A}, \tau) \cap L^\infty(\mathcal{A}, \tau)$  then  $x = 0$ .

### 1.8 Proposition. (Riesz-Thorin-Kunze interpolation)

Let  $T : \mathcal{S} \rightarrow \tilde{\mathcal{A}}$  be a linear map satisfying

$$\|Tx\|_{q_i} \leq M_i \|x\|_{p_i}, \quad \forall x \in \mathcal{S}, \quad i = 1, 2,$$

with  $p_i, q_i \in [1, \infty]$  and  $M_i > 0$ . If  $\frac{1}{p} := \frac{t}{p_1} + \frac{1-t}{p_2}, \frac{1}{q} := \frac{t}{q_1} + \frac{1-t}{q_2}$ , where  $t \in (0, 1)$ , then,  $\forall x \in \mathcal{S}$ , we get

$$\|Tx\|_q \leq M_1^{1-t} M_2^t \|x\|_p.$$

## Section 2. Markov semigroups and Dirichlet forms.

In this section we give the basic definitions and prove the main theorems which constitute the basis of the theory of non symmetric Dirichlet forms in a non commutative setting. In our exposition we generalize to the non abelian case results and techniques of Chap. I, Sec. 4 in [MR]. In particular, the classical space of square integrable functions on a measure space is replaced by the space of the operators affiliated to a von Neumann algebra  $\mathcal{A}$  which are square integrable w.r.t. a normal, semifinite, faithful trace  $\tau$ .

**2.1 Definition.** (i) A bounded linear operator  $G$  on  $L^2(\mathcal{A}, \tau)$  is called *sub-Markovian* if

$$0 \leq x \leq 1 \Rightarrow 0 \leq Gx \leq 1, \quad \forall x \in L^2(\mathcal{A}, \tau).$$

A strongly continuous contraction resolvent  $\{G_\alpha\}_{\alpha>0}$ , resp. semigroup  $\{T_t\}_{t>0}$ , is called *sub-Markovian* if all  $\alpha G_\alpha$ ,  $\alpha > 0$ , resp.  $T_t$ ,  $t > 0$ , are sub-Markovian.

(ii) A closed densely defined operator  $\{L, \mathcal{D}(L)\}$  on  $L^2(\mathcal{A}, \tau)$  is called *Dirichlet operator* if  $(Lx, (x - 1)^+) \leq 0$  for each  $x \in \mathcal{D}(L)_h$ .

(iii) A coercive closed form on  $L^2(\mathcal{A}, \tau)$  is called a *Dirichlet form* if, for all  $x \in \mathcal{D}(\mathcal{E})_h$ ,  $x^+ \wedge 1 \in \mathcal{D}(\mathcal{E})$  and

$$\begin{aligned} \mathcal{E}(x - x^+ \wedge 1, x + x^+ \wedge 1) &\geq 0 \\ \mathcal{E}(x + x^+ \wedge 1, x - x^+ \wedge 1) &\geq 0 \end{aligned} \tag{2.1}$$

If only the first inequality in (2.1) holds, the form is called *1/2-Dirichlet*.

As in the classical case, if the form  $\mathcal{E}$  is symmetric each of the two inequalities in (2.1) is equivalent to the usual definition of Dirichlet form (see e.g. [AH]).

The following two theorems state the equivalence among the objects described in Definition 2.1.

**2.2 Theorem.** Let  $\{\mathcal{E}, \mathcal{D}(\mathcal{E})\}$  be a coercive closed form on  $L^2(\mathcal{A}, \tau)$  with corresponding semigroup  $\{T_t\}_{t>0}$ , resolvent  $\{G_\alpha\}_{\alpha>0}$  and generator  $\{L, \mathcal{D}(L)\}$ . Then the following are equivalent:

- (a) The form  $\mathcal{E}$  is 1/2-Dirichlet.
- (b) The semigroup  $\{T_t\}_{t>0}$  is sub-Markovian.
- (c) The resolvent  $\{G_\alpha\}_{\alpha>0}$  is sub-Markovian.
- (d) The generator  $\{L, \mathcal{D}(L)\}$  is a Dirichlet operator.

**2.3 Theorem.** Under the same hypotheses of the preceding theorem, the following are equivalent:

- (a) The form  $\mathcal{E}$  is Dirichlet.
- (b) The semigroups  $\{T_t\}_{t>0}$  and  $\{T_t^*\}_{t>0}$  are sub-Markovian.



- (c) The resolvents  $\{G_\alpha\}_{\alpha>0}$  and  $\{G_\alpha^*\}_{\alpha>0}$  are sub-Markovian.
- (d) The generators  $L$  and  $L^*$  are Dirichlet operators.

The proof of the preceding theorems follows directly from propositions 2.6 and 2.7.

**2.4 Lemma.** *A bounded linear operator  $G$  on  $L^2(\mathcal{A}, \tau)$  is sub-Markovian iff*

$$\begin{cases} x \geq 0 \Rightarrow Gx \geq 0 \\ x \leq 1 \Rightarrow Gx \leq 1 \end{cases} \quad \forall x \in L^2(\mathcal{A}, \tau)$$

**Proof.** Sufficiency is true by definition. Now let  $x \in L^2(\mathcal{A}, \tau)$ ,  $x \geq 0$ , and define  $x_n := x \wedge n$ . Clearly  $x_n \rightarrow x$  in  $L^2(\mathcal{A}, \tau)$ , and therefore  $Gx_n \rightarrow Gx$  in  $L^2(\mathcal{A}, \tau)$  by continuity. Moreover  $0 \leq \frac{x_n}{n} \leq 1$  which implies  $0 \leq G(\frac{x_n}{n}) \leq 1$ , and therefore  $Gx_n \geq 0$ . Then, since the positive part of  $L^2(\mathcal{A}, \tau)$  is closed, we get  $Gx \geq 0$ . Finally let  $x \in L^2(\mathcal{A}, \tau)$ ,  $x \leq 1$ . If  $x = x^+ - x^-$  is the decomposition of  $x$  into positive and negative part, we have  $0 \leq x^+ \leq 1$ ,  $0 \leq x^-$  and therefore  $0 \leq Gx^+ \leq 1$  by the sub-Markov property and  $0 \leq Gx^-$  by the first part of this theorem. Then the thesis follows by linearity.  $\square$

**2.5 Lemma.** *Let  $\{x_n\}$  be a sequence converging to  $x$  in  $L^2(\mathcal{A}, \tau)$  for which  $0 \leq x_n \leq 1$ ,  $\forall n \in \mathbf{N}$ . Then  $0 \leq x \leq 1$ .*

**Proof.** The fact that  $x \geq 0$  follows because  $L^2(\mathcal{A}, \tau)_+$  is norm closed. Moreover, since  $x_n$  converges weakly in  $L^2$  and is uniformly bounded in  $L^\infty(\mathcal{A}, \tau)$ ,  $x_n$  converges to  $x$  weak\* in  $L^\infty(\mathcal{A}, \tau)$  and therefore  $\|x\|_\infty \leq 1$ . This implies  $x \leq 1$  because  $x$  is positive.  $\square$

**2.6 Proposition.** *Let  $\{G_\alpha\}_{\alpha>0}$  be a strongly continuous contraction resolvent on  $L^2(\mathcal{A}, \tau)$  with corresponding generator  $L$  and semigroup  $\{T_t\}_{t>0}$ . Then the following are equivalent:*

- (i)  $\{G_\alpha\}_{\alpha>0}$  is sub-Markovian.
- (ii)  $\{T_t\}_{t>0}$  is sub-Markovian.
- (iii)  $L$  is a Dirichlet operator.

**Proof.** (i)  $\Rightarrow$  (ii): Let  $x \in L^2(\mathcal{A}, \tau)$  and  $0 \leq x \leq 1$ . Then, for all  $\beta > 0$ ,  $x_\beta := \beta G_\beta x$  is in  $\mathcal{D}(L)$  and  $0 \leq x_\beta \leq 1$  since the resolvent  $G_\beta$  is sub-Markovian, therefore, by formula (1.1) and lemma 2.5,  $0 \leq T_t x_\beta \leq 1$ . Moreover  $x_\beta \rightarrow x$  in  $L^2(\mathcal{A}, \tau)$  when  $\beta \rightarrow \infty$ , therefore, again by lemma 2.5,  $0 \leq T_t x \leq 1$ , i.e.  $T_t$  is sub-Markovian.

(ii)  $\Rightarrow$  (iii): Let  $x \in L^2(\mathcal{A}, \tau)_h$ . Then

$$((x-1)^+, T_t(x-1)^+) \leq ((x-1)^+, (x-1)^+) = ((x-1)^+, (x-1))$$

by the Schwartz inequality and the fact that  $(x-1)^+$  and  $(x-1)^-$  are orthogonal in  $L^2(\mathcal{A}, \tau)$ . Moreover  $T_t(x \wedge 1) \leq 1$  by lemma 2.4. Therefore, since  $x = (x-1)^+ + x \wedge 1$ , we have

$$\begin{aligned} ((x-1)^+, T_t x) &= ((x-1)^+, T_t(x-1)^+) + ((x-1)^+, T_t(x \wedge 1)) \\ &\leq ((x-1)^+, (x-1)) + \tau((x-1)^+) \\ &= ((x-1)^+, x). \end{aligned}$$

Therefore we get

$$((x-1)^+, Lx) = \lim_{t \downarrow 0} \frac{1}{t} ((x-1)^+, T_t x - x) \leq 0, \quad \forall x \in \mathcal{D}(L).$$

(iii)  $\Rightarrow$  (i). Let  $x \in L^2(\mathcal{A}, \tau)_h$ ,  $\alpha > 0$  and  $y := \alpha G_\alpha x$ . We want to prove that if  $0 \leq x \leq 1$  then  $0 \leq y \leq 1$ . Indeed, for  $x \leq 1$ , we have

$$\begin{aligned} \alpha((y-1)^+, y) &= ((y-1)^+, \alpha y - Ly) + ((y-1)^+, Ly) \\ &\leq \alpha((y-1)^+, x) \leq \alpha \tau((y-1)^+). \end{aligned}$$

As a consequence,

$$\|(y-1)^+\|_2 = ((y-1)^+, y) - \tau((y-1)^+) \leq 0,$$

hence  $y \leq 1$ . On the other hand, if  $x \geq 0$ , then  $-nx \leq 1 \ \forall n \in \mathbf{N}$ , therefore, by the previous result,  $-ny \leq 1, \forall n \in \mathbf{N}$ , i.e.  $y \geq 0$ .  $\square$

**2.7 Proposition.** Let  $\{\mathcal{E}, \mathcal{D}(\mathcal{E})\}$  be a coercive closed form on  $L^2(\mathcal{A}, \tau)$  with resolvent  $\{G_\alpha\}_{\alpha>0}$ . Then the following are equivalent:

- (i) For all  $x \in \mathcal{D}(\mathcal{E})_h$  and  $\alpha \geq 0$ ,  $x \wedge \alpha \in \mathcal{D}(\mathcal{E})$  and  $\mathcal{E}(x - x \wedge \alpha, x \wedge \alpha) \geq 0$ .
- (ii) For all  $x \in \mathcal{D}(\mathcal{E})_h$ ,  $x^+ \wedge 1 \in \mathcal{D}(\mathcal{E})$  and  $\mathcal{E}(x - x^+ \wedge 1, x^+ \wedge 1) \geq 0$ .
- (iii)  $\mathcal{E}$  is a 1/2-Dirichlet form.
- (iv)  $\{G_\alpha\}_{\alpha>0}$  is sub-Markovian.

The analogous equivalences hold when  $\{G_\alpha\}_{\alpha>0}$  is replaced by its adjoint and  $\mathcal{E}$  by the form  $\mathcal{E}^\dagger(x, y) := \mathcal{E}(y, x)$ .

**Proof.** (i)  $\Rightarrow$  (ii). Let  $x \in \mathcal{D}(\mathcal{E})_h$ , then, by (i), we get  $x^-, x^+, x^+ \wedge 1 \in \mathcal{D}(\mathcal{E})$ . As a consequence

$$\begin{aligned} \mathcal{E}(x - x^+ \wedge 1, x^+ \wedge 1) &= \mathcal{E}(x^+ - x^+ \wedge 1, x^+ \wedge 1) - \mathcal{E}(x^-, x^+ \wedge 1) \\ &\geq -\mathcal{E}((x \wedge 1)^-, (x \wedge 1)^+). \end{aligned}$$

Now for any  $y \in \mathcal{D}(\mathcal{E})_h$  we have, again by (i),

$$\mathcal{E}(y^-, y^+) = \mathcal{E}(y^+ - y, y^+) = -\mathcal{E}((-y) - (-y) \wedge 0, (-y) \wedge 0) \leq 0$$

therefore  $\mathcal{E}(x - x^+ \wedge 1, x^+ \wedge 1) \geq 0$ .

(ii)  $\Rightarrow$  (iii). Since  $\mathcal{E}$  is a real-positive sesquilinear form and (ii) holds, we get, for all  $x \in \mathcal{D}(\mathcal{E})_h$ ,

$$\mathcal{E}(x - x^+ \wedge 1, x + x^+ \wedge 1) = \mathcal{E}(x - x^+ \wedge 1, x - x^+ \wedge 1) + 2\mathcal{E}(x - x^+ \wedge 1, x^+ \wedge 1) \geq 0$$

(iii)  $\Rightarrow$  (iv). Let  $y \in L^2(\mathcal{A}, \tau)$ ,  $0 \leq y \leq 1$ . We have to show that  $x := \alpha G_\alpha y$  satisfies  $0 \leq x \leq 1$ . Indeed

$$\begin{aligned} & \|x - x^+ \wedge 1\|^2 + (x - x^+ \wedge 1, x^+ \wedge 1 - y) = \\ & = (x - x^+ \wedge 1, x - y) \\ & = -\frac{1}{\alpha} \mathcal{E}(x - x^+ \wedge 1, x) \\ & = -\frac{1}{2\alpha} (\mathcal{E}(x - x^+ \wedge 1, x + x^+ \wedge 1) + \mathcal{E}(x - x^+ \wedge 1, x - x^+ \wedge 1)) \\ & \leq 0 \end{aligned} \tag{2.2}$$

where the equality in the second line follows from theorem 1.4. Let us introduce the functions  $f, g, h : \mathbf{R} \rightarrow \mathbf{R}$ ,

$$\begin{aligned} f(t) &= t\chi_{(-\infty, 0]}(t), \\ g(t) &= (t \wedge 1)\chi_{[0, \infty)}(t), \\ h(t) &= (t - 1)\chi_{[1, \infty)}(t). \end{aligned}$$

Then  $f g \equiv 0$ ,  $g h \equiv h$ ,  $g(x) = x^+ \wedge 1$  and  $x - g(x) = f(x) + h(x)$ . Therefore

$$\begin{aligned} (x - x^+ \wedge 1, x^+ \wedge 1 - y) &= \tau((x - x^+ \wedge 1)(x^+ \wedge 1 - y)) \\ &= \tau(f(x)(g(x) - y)) + \tau(h(x)(g(x) - y)) \\ &= \tau((-f(x))y) + \tau(h(x)(1 - y)) \geq 0 \end{aligned} \tag{2.3}$$

where we used proposition 1.6(vi). Finally equations (2.2) and (2.3) imply  $\|x - x^+ \wedge 1\| = 0$ , i.e.  $0 \leq x \leq 1$ .

(iv)  $\Rightarrow$  (i). Let  $x \in \mathcal{D}(\mathcal{E})_h$ ,  $\alpha \geq 0$ . Now we prove that  $x \wedge \alpha \in \mathcal{D}(\mathcal{E})$ : since  $x = (x - \alpha)^+ + x \wedge \alpha$ , it suffices to prove  $(x - \alpha)^+ \in \mathcal{D}(\mathcal{E})$ . Recalling that, by proposition 1.5,  $\mathcal{E}^{(\beta)}(y, z) = \beta \tau(y^*(z - \beta G_\beta z))$ , for  $y, z \in L^2(\mathcal{A}, \tau)$ , we have

$$\begin{aligned} \mathcal{E}^{(\beta)}((x - \alpha)^+, x \wedge \alpha) &= \beta \tau((x - \alpha)^+(x \wedge \alpha)) - \beta \tau((x - \alpha)^+ \beta G_\beta (x \wedge \alpha)) \\ &\geq \alpha \beta \tau((x - \alpha)^+) - \alpha \beta \tau((x - \alpha)^+) = 0 \end{aligned} \tag{2.4}$$

where, since  $x \wedge \alpha \leq \alpha$ , the inequality in (2.4) follows from lemma 2.4, proposition 1.6(vi) and the fact that  $\beta G_\beta$  is sub-Markovian.

Therefore,

$$\begin{aligned}
\mathcal{E}_1^{(\beta)}((x - \alpha)^+, (x - \alpha)^+) &= \mathcal{E}_1^{(\beta)}((x - \alpha)^+, x - x \wedge \alpha) \\
&= \mathcal{E}_1^{(\beta)}((x - \alpha)^+, x) - \mathcal{E}_1^{(\beta)}((x - \alpha)^+, x \wedge \alpha) \\
&= \mathcal{E}_1^{(\beta)}((x - \alpha)^+, x) - \mathcal{E}^{(\beta)}((x - \alpha)^+, x \wedge \alpha) - ((x - \alpha)^+, x \wedge \alpha) \\
&\leq \mathcal{E}_1^{(\beta)}((x - \alpha)^+, x) \\
&\leq (K + 1)\mathcal{E}_1(x, x)^{1/2}\mathcal{E}_1^{(\beta)}((x - \alpha)^+, (x - \alpha)^+)^{1/2}
\end{aligned}$$

where the last inequality follows from proposition 1.5(i). As a consequence,

$$\mathcal{E}^{(\beta)}((x - \alpha)^+, (x - \alpha)^+) \leq \mathcal{E}_1^{(\beta)}((x - \alpha)^+, (x - \alpha)^+) \leq (K + 1)^2 \mathcal{E}_1(x, x). \quad (2.5)$$

Now proposition 1.5(ii) and (2.5) imply  $(x - \alpha)^+ \in \mathcal{D}(\mathcal{E})$ .

Finally we prove that  $\mathcal{E}(x - x \wedge \alpha, x \wedge \alpha) \geq 0$ : we have

$$\mathcal{E}^{(\beta)}(x - x \wedge \alpha, x \wedge \alpha) = \mathcal{E}^{(\beta)}((x - \alpha)^+, x \wedge \alpha) \geq 0$$

by (2.4), hence the result follows by proposition 1.5(iii).  $\square$

We conclude this section with a theorem in which it is shown that a smooth version of the definition of a Dirichlet form can be given. More precisely, the *normal contraction*  $x^+ \wedge 1$  in (2.1) may be substituted by a family of  $\mathcal{C}^\infty$  contractions.

**2.8 Theorem.** *Let  $\{\mathcal{E}, \mathcal{D}(\mathcal{E})\}$  be a coercive closed form on  $L^2(\mathcal{A}, \tau)$ . Then, the following are equivalent:*

- (i)  $\mathcal{E}$  is a Dirichlet form.
- (ii) For each  $x \in \mathcal{D}(\mathcal{E})_h$  there exists a family of functions  $\varphi_\varepsilon : \mathbf{R} \rightarrow [-\varepsilon, 1 + \varepsilon]$ ,  $\varepsilon \geq 0$ , such that

- (a)  $\varphi_\varepsilon(t) = t$  for all  $t \in [0, 1]$ .
- (b)  $\varphi_\varepsilon$  is Lipschitz continuous with Lipschitz constant 1.
- (c)  $\varphi_\varepsilon(x) \in \mathcal{D}(\mathcal{E})$
- (d)  $\liminf_{\varepsilon \rightarrow 0} \mathcal{E}(x \mp \varphi_\varepsilon(x), x \pm \varphi_\varepsilon(x)) \geq 0$

**Proof.** The implication (i)  $\Rightarrow$  (ii) is trivial.

(ii)  $\Rightarrow$  (i). Let us show that

$$\lim_{\varepsilon \rightarrow 0} \|\varphi_\varepsilon(x) - x^+ \wedge 1\|_2 = 0. \quad (2.6)$$

Indeed, setting  $\varphi_0(t) := (t \vee 0) \wedge 1$ , we have

$$(\varphi_\varepsilon(x) - \varphi_0(x))^2 = \int_{-\infty}^{+\infty} (\varphi_\varepsilon(\lambda) - \varphi_0(\lambda))^2 de_\lambda$$

where  $e_\lambda$  is the spectral family associated with  $x$ , therefore

$$\begin{aligned} & \tau((\varphi_\varepsilon(x) - \varphi_0(x))^2) \\ &= \tau \left( \int_{-\infty}^{-\sqrt{\varepsilon}} (\varphi_\varepsilon(\lambda))^2 de_\lambda + \int_{-\sqrt{\varepsilon}}^0 (\varphi_\varepsilon(\lambda))^2 de_\lambda + \int_1^{+\infty} (\varphi_\varepsilon(\lambda) - 1)^2 de_\lambda \right) \\ &\leq \varepsilon^2 \tau(\chi_{(-\infty, -\sqrt{\varepsilon}]}(x)) + \tau(x^2 \chi_{[-\sqrt{\varepsilon}, 0]}(x)) + \varepsilon^2 \tau(\chi_{[1, +\infty)}(x)). \end{aligned}$$

Then (2.6) follows by the following:

$$\begin{aligned} \tau(\chi_{[1, +\infty)}(x)) &\leq \tau(x^2 \chi_{[1, +\infty)}(x)) \leq \tau(x^2) < \infty, \\ \tau(\chi_{(-\infty, -\sqrt{\varepsilon}]}(x)) &\leq \tau \left( \int_{-\infty}^{-\sqrt{\varepsilon}} \frac{\lambda^2}{\varepsilon} de_\lambda \right) \leq \frac{1}{\varepsilon} \tau(x^2), \end{aligned}$$

and

$$\tau(x^2 \chi_{[-\sqrt{\varepsilon}, 0]}(x)) = \tau \left( \int_{-\sqrt{\varepsilon}}^0 \lambda^2 de_\lambda \right) \rightarrow 0,$$

when  $\varepsilon \rightarrow 0$ , because  $\mu(E) := \tau(\int_E \lambda^2 de_\lambda)$  is a finite measure and  $\mu(\{0\}) = 0$ .

Summing the two inequalities in (d) it follows

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}(\varphi_\varepsilon(x), \varphi_\varepsilon(x)) \leq \mathcal{E}(x, x)$$

therefore, applying [MR, proposition 2.12] to  $\varphi_\varepsilon(x)$  and the form  $\mathcal{E}$ , we get a sequence  $\varepsilon_n \rightarrow 0$  such that  $\varphi_{\varepsilon_n}(x)$  converges weakly in  $\{\mathcal{D}(\mathcal{E}), \tilde{\mathcal{E}}_1\}$  to  $x^+ \wedge 1$ , so that  $x^+ \wedge 1 \in \mathcal{D}(\mathcal{E})$ , and

$$\mathcal{E}(x^+ \wedge 1, x^+ \wedge 1) \leq \liminf_{n \rightarrow \infty} \mathcal{E}(\varphi_{\varepsilon_n}(x), \varphi_{\varepsilon_n}(x))$$

Moreover, by the weak sector condition, the functional  $\mathcal{E}_1(\cdot, x)$  is continuous in  $\{\mathcal{D}(\mathcal{E}), \tilde{\mathcal{E}}_1\}$ , therefore

$$\lim_{n \rightarrow \infty} \mathcal{E}(x, \varphi_{\varepsilon_n}(x)) = \mathcal{E}(x, x^+ \wedge 1).$$

Finally, we have

$$\begin{aligned} \mathcal{E}(x \pm (x^+ \wedge 1), x \mp (x^+ \wedge 1)) &\geq \mathcal{E}(x, x) \mp \lim_{n \rightarrow \infty} \mathcal{E}(x, \varphi_{\varepsilon_n}(x)) \\ &\quad \pm \lim_{n \rightarrow \infty} \mathcal{E}(\varphi_{\varepsilon_n}(x), x) - \liminf_{n \rightarrow \infty} \mathcal{E}(\varphi_{\varepsilon_n}(x), \varphi_{\varepsilon_n}(x)) = \\ &= \limsup_{n \rightarrow \infty} \mathcal{E}(x \pm \varphi_{\varepsilon_n}(x), x \mp \varphi_{\varepsilon_n}(x)) \geq 0 \end{aligned}$$

where the last inequality follows by hypothesis (d).  $\square$

### Section 3. $L^p$ extensions of sub-Markovian semigroups and complete positivity.

This section is devoted to the extension of some properties of sub-Markovian semigroups, already studied in [DL], to the non symmetric context. In particular we prove that sub-Markovian semigroups may be extended to  $L^p$  spaces and study a class of sub-Markovian semigroups on  $L^\infty(\mathcal{A}, \tau)$ , showing a correspondence between such semigroups and those on  $L^2(\mathcal{A}, \tau)$ . Finally we exploit the consequences of complete positivity for semigroups and Dirichlet forms such as the contraction property for semigroups on  $L^\infty(\mathcal{A}, \tau)$ .

#### 3.1 Definition.

- (i) Let  $M \in \mathcal{B}(L^p(\mathcal{A}, \tau))$ , then we define  ${}^tM \in \mathcal{B}(L^{p'}(\mathcal{A}, \tau))$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ , as the unique linear operator satisfying  $({}^tMx, y) = (x, My)$ ,  $\forall x \in L^p, y \in L^{p'}$ .
- (ii)  $M \in \mathcal{B}(L^p(\mathcal{A}, \tau))$ ,  $p \in [1, \infty)$ , is said a sub-Markovian operator on  $L^p$  if  $x \in L^p(\mathcal{A}, \tau)$ ,  $0 \leq x \leq 1 \Rightarrow 0 \leq Mx \leq 1$ .
- (iii)  $M \in \mathcal{B}(L^\infty(\mathcal{A}, \tau))$ , is said a sub-Markovian operator on  $L^\infty$  if it is weak\* continuous and  $0 \leq x \leq 1 \Rightarrow 0 \leq Mx \leq 1$ .
- (iv)  $\{T_t\}_{t \geq 0} \subset \mathcal{B}(L^p(\mathcal{A}, \tau))$ ,  $p \in [1, \infty)$ , is said a sub-Markovian semigroup on  $L^p$ , if  $T_t$  is a sub-Markovian operator on  $L^p$ , for all  $t > 0$ , and  $T_t \rightarrow I$ ,  $t \rightarrow 0$ , strongly on  $L^p(\mathcal{A}, \tau)$ .
- (v)  $\{T_t\}_{t \geq 0} \subset \mathcal{B}(L^\infty(\mathcal{A}, \tau))$  is said a sub-Markovian semigroup on  $L^\infty$ , if  $T_t$  is a sub-Markovian operator on  $L^\infty$ , for all  $t > 0$ , and  $T_t \rightarrow I$ ,  $t \rightarrow 0$ , weak\* on  $L^\infty(\mathcal{A}, \tau)$ .

**3.2 Remark.** Notice that the definition of sub-Markovian operator on  $L^2$  differs from that in the preceding sections in that we do not require contractivity here.

**3.3 Proposition.** Let  $M$  and  $M^*$  be sub-Markovian operators on  $L^2(\mathcal{A}, \tau)$ . Then:

- (i)  $\|Mx\|_p \leq 2\|x\|_p$ ,  $\|M^*x\|_p \leq 2\|x\|_p$ ,  $x \in L^p \cap L^2$ ,  $p \in [1, \infty]$ .  
Let now  $M^{(p)}$ , resp.  $M^{*(p)}$ , be the unique continuous extensions of  $M|_{L^p \cap L^2}$ , resp.  $M^*|_{L^p \cap L^2}$ , to  $L^p$ , for  $p \in [1, \infty)$ . Then:
- (ii)  ${}^t(M^{(p)}) = M^{*(p')}$ , resp.  ${}^t(M^{*(p)}) = M^{(p')}$ , for  $p \in (1, \infty)$ , and  $\frac{1}{p} + \frac{1}{p'} = 1$ .  
Set  $M^{(\infty)} := {}^t(M^{*(1)})$ , resp.  $M^{*(\infty)} := {}^t(M^{(1)})$ . Then:
- (iii)  $M^{(p)}x = M^{(q)}x$ , resp.  $M^{*(p)}x = M^{*(q)}x$ , for  $x \in L^p \cap L^q$ ,  $p, q \in [1, \infty]$ .

**Proof.** (i) Let  $x \in (L^\infty \cap L^2)_h$ , then  $\|x\|_\infty \leq 1 \iff 0 \leq x_\pm \leq 1 \Rightarrow 0 \leq Mx_\pm \leq 1$  so that  $\|Mx\|_\infty \leq \|x\|_\infty$ . Let now  $z \in L^\infty \cap L^2$  and  $z = x + iy$ , with  $x, y \in (L^\infty \cap L^2)_h$ ; then  $\|Mz\|_\infty \leq \|Mx\|_\infty + \|My\|_\infty \leq \|x\|_\infty + \|y\|_\infty \leq 2\|z\|_\infty$ . In the same way  $\|M^*z\|_\infty \leq 2\|z\|_\infty$ ,  $z \in L^\infty \cap L^2$ .

Suppose now  $x \in L^1 \cap L^2$ , as  $B_0 := \{y \in L^2 \cap L^\infty : \|y\|_\infty \leq 1\}$  is weak\* dense in the unit ball of  $L^\infty$ , we have

$$\begin{aligned}\|Mx\|_1 &= \sup\{|(y, Mx)| : y \in B_0\} \\ &= \sup\{|(M^*y, x)| : y \in B_0\} \\ &\leq \sup\{\|M^*y\|_\infty : y \in B_0\} \|x\|_1 \leq 2\|x\|_1.\end{aligned}$$

The same holds for  $M^*$ .

By Riesz-Thorin-Kunze interpolation (proposition 1.8) we have  $\|Mx\|_p \leq 2\|x\|_p$ , for  $x \in L^p \cap L^2$ , and analogously for  $M^*$ .

(ii) Let  $p \in (1, \infty)$ ,  $x \in L_h^p$ ,  $y \in L^1 \cap L^\infty$ , and  $e_n := e_{|x|}((\frac{1}{n}, n))$ ; then  $xe_n \rightarrow x$  in  $L^p$ , by proposition 1.7(ii), and  $xe_n \in (L^p \cap L^2)_h$ , so that we have

$$\begin{aligned}(x, {}^t(M^{(p)})y) &= (M^{(p)}x, y) = \lim_{n \rightarrow \infty} (M^{(p)}(xe_n), y) = \lim_{n \rightarrow \infty} (M(xe_n), y) \\ &= \lim_{n \rightarrow \infty} (xe_n, M^*y) = \lim_{n \rightarrow \infty} (xe_n, M^{*(p')}y) = (x, M^{*(p')}y).\end{aligned}$$

Hence the thesis, by linearity and the density of  $L^1 \cap L^\infty$  in  $L^{p'}$ .

(iii) Suppose that  $p < q < \infty$  and  $x \in (L^p \cap L^q)_h$ . Then we have

$$M^{(p)}x = \lim_{n \rightarrow \infty} M^{(p)}(xe_n) = \lim_{n \rightarrow \infty} M^{(q)}(xe_n) = M^{(q)}x,$$

and, by linearity, we are through.

Now consider the case  $q = \infty$  and let  $x \in (L^p \cap L^\infty)_h$  and  $y \in L^1 \cap L^\infty$ , then we have

$$\begin{aligned}(y, M^{(p)}x) &= \lim_{n \rightarrow \infty} (y, M^{(p)}(xe_n)) = \lim_{n \rightarrow \infty} (y, M(xe_n)) \\ &= \lim_{n \rightarrow \infty} (M^*y, xe_n) = (M^{*(1)}y, x) = (y, M^{(\infty)}x).\end{aligned}$$

So, from proposition 1.7(iii),  $M^{(p)}x = M^{(\infty)}x$ , and, by linearity, we are through.  $\square$

**3.4 Corollary.** *Let  $M$  and  $M^*$  be sub-Markovian operators on  $L^2$  and  $M^{(p)}$ ,  $M^{*(p)}$  their extensions to  $L^p$ ,  $p \in [1, \infty]$ . Then  $M^{(p)}$ ,  $M^{*(p)}$  are sub-Markovian operators on  $L^p$ .*

**Proof.** Let us consider first the case  $p < \infty$ . Let  $x \in L^p$ ,  $0 \leq x \leq 1$ , and observe that  $x_n := xe_x((\frac{1}{n}, 1)) \in L^2 \cap L^p$ ,  $0 \leq x_n \leq 1$  and  $x_n \rightarrow x$  in  $L^p$ . Therefore  $0 \leq M^{(p)}x_n \leq 1$  and  $M^{(p)}x_n \rightarrow M^{(p)}x$  in  $L^p$ . From the following lemma it follows that  $0 \leq M^{(p)}x \leq 1$ , that is  $M^{(p)}$  is a sub-Markovian operator on  $L^p$ .

Now consider the case  $p = \infty$ , and observe that  $M^{(\infty)}$  is obviously weak\* continuous. Besides the \*-algebra  $L^2 \cap L^\infty$  is strongly dense in  $L^\infty$  so that  $\forall x \in L^\infty$ ,  $0 \leq x \leq 1$  there exists, by Kaplansky's density theorem, a net  $\{x_\alpha\} \subset L^2 \cap L^\infty$  s.t.  $0 \leq x_\alpha \leq 1$  and  $x_\alpha \rightarrow x$  strongly hence  $\sigma$ -weakly. Then from  $M^{(\infty)}x_\alpha \rightarrow M^{(\infty)}x$   $\sigma$ -weakly and  $0 \leq M^{(\infty)}x_\alpha \leq 1$  it follows  $0 \leq M^{(\infty)}x \leq 1$ , that is  $M^{(\infty)}$  is sub-Markovian. A similar proof works also for  $M^{*(p)}$ .  $\square$

**3.5 Lemma.** *Let  $x \in L^p(\mathcal{A}, \tau)$ ,  $p \in [1, \infty)$  and  $\{x_n\}$  be s.t.  $0 \leq x_n \leq 1$  and  $x_n \rightarrow x$  in  $L^p$ . Then  $0 \leq x \leq 1$*

**Proof.** Let  $\{x_{n_k}\}$  be s.t.  $x_{n_k} \rightarrow y \in L^\infty(\mathcal{A}, \tau)$  weak\*, so that  $0 \leq y \leq 1$ . Then  $\forall z \in L^1 \cap L^\infty$  we get  $(z, y) = \lim_{k \rightarrow \infty} (z, x_{n_k}) = (z, x)$  and by proposition 1.7(iii) we are through.  $\square$

**3.6 Theorem.** *Let  $\{T_t\}_{t \geq 0}$ ,  $\{T_t^*\}_{t \geq 0}$  be sub-Markovian semigroups on  $L^2$ . Then their extensions to  $L^p$  are sub-Markovian semigroups on  $L^p$ , for  $p \in [1, \infty]$ .*

**Proof.** By proposition 3.3,  $\|T_t\|_p \leq 2$ ,  $t \geq 0$ ,  $p \in [1, \infty]$ . Let  $p \in (1, \infty)$ ,  $p'$  the conjugate exponent,  $x \in L^{p'} \cap L^2$ ,  $y \in L^p \cap L^2$ , then we have

$$(x, (T_t^{(p)} - I)y) = (x, (T_t - I)y) = ((T_t^* - I)x, y) \rightarrow 0, t \rightarrow 0$$

so that, from the density of  $L^{p'} \cap L^2$  in  $L^{p'}$  and of  $L^p \cap L^2$  in  $L^p$ , we get the weak continuity of  $\{T_t\}_{t \geq 0}$  on  $L^p$ , and, from [D1, proposition 1.23], the strong continuity. Let now  $p = 1$  and  $e \in \text{Proj}(\mathcal{A}) \cap L^1$ , then we have

$$\|T_t^{(1)}e - e\|_1 = \|T_te - e\|_1 \leq \|e\|_2 \|T_te - e\|_2 \rightarrow 0, t \rightarrow 0.$$

As  $\{x = \sum_{i=1}^n \lambda_i e_i : \lambda_i > 0, e_i \in \text{Proj}(\mathcal{A}) \cap L^1\}$  is total in  $L^1$ , by proposition 1.7(i), we are through.

Now let us observe that the same proof also works for  $\{T_t^*\}_{t \geq 0}$ , so  $\{T_t^{*(p)}\}_{t \geq 0}$ ,  $p \in [1, \infty)$ , is a sub-Markovian semigroup on  $L^p$ .

Finally let  $p = \infty$ ,  $x \in L^\infty$  and  $y \in L^1 \cap L^2$ ; then we have  $(y, (T_t^{(\infty)} - I)x) = ((T_t^{*(1)} - I)y, x) \rightarrow 0, t \rightarrow 0$ , as we have already proved, so that  $\{T_t^{(\infty)}\}_{t \geq 0}$  is a weak\* continuous semigroup on  $L^\infty$ . Analogously for  $\{T_t^{*(\infty)}\}_{t \geq 0}$ .  $\square$

Let us now show there is a converse of the preceding theorem in case  $p = \infty$ .

**3.7 Theorem.** *Let  $\{T_t\}_{t \geq 0}$ ,  $\{\hat{T}_t\}_{t \geq 0}$  be sub-Markovian semigroups on  $\{\mathcal{A}, \tau\}$  s.t.*

$$\tau(x(T_ty)) = \tau((\hat{T}_tx)y), x, y \in L^1 \cap L^\infty.$$



Then  $\{T_t\}_{t \geq 0}$  and  $\{\hat{T}_t\}_{t \geq 0}$  are the unique weak\*-continuous extensions of sub-Markovian semigroups on  $L^2$  which are adjoint to each other.

**Proof.** Fix  $t \geq 0$  and write  $T$  for  $T_t$  and  $\hat{T}$  for  $\hat{T}_t$ . As  $T$  and  $\hat{T}$  are sub-Markovian we get  $\|T\|_\infty \leq 2$ ,  $\|\hat{T}\|_\infty \leq 2$ . Define  $T_* : L^1 \rightarrow L^1$  and  $\hat{T}_* : L^1 \rightarrow L^1$  by  $\tau((\hat{T}_*x)y) := \tau(x(Ty))$  and  $\tau((T_*x)y) := \tau(x(\hat{T}y))$ ,  $x \in L^1$ ,  $y \in L^\infty$ . Then  $T_*$ ,  $\hat{T}_*$  are positivity preserving as  $x \in L^1_+ \Rightarrow \tau((\hat{T}_*x)y) = \tau(x(Ty)) \geq 0$ ,  $\forall y \in L^\infty_+$ , that is  $\hat{T}_*x \geq 0$  and analogously  $T_*x \geq 0$ . Moreover  $\|\hat{T}_*\|_1 = \|T\|_\infty$  and  $\|T_*\|_1 = \|\hat{T}\|_\infty$ . Besides,  $\forall x, y \in L^1 \cap L^\infty$ , we get  $\tau((\hat{T}_*x)y) = \tau(x(Ty)) = \tau((\hat{T}x)y)$  that is  $\hat{T}_*x = \hat{T}x$  and analogously  $T_*x = Tx$  for all  $x \in L^1 \cap L^\infty$ . Therefore, by Riesz-Thorin-Kunze interpolation (proposition 1.8),  $T$  and  $\hat{T}$  extend uniquely from  $L^1 \cap L^\infty$  to  $L^2$  with norm no greater than 2, and  $\tau(x(Ty)) = \tau((\hat{T}x)y)$ ,  $x, y \in L^2$ .

Therefore  $\{T_t\}_{t \geq 0}$  and  $\{\hat{T}_t\}_{t \geq 0}$  extend uniquely to  $L^2$ . Let now  $x, y \in L^2$ , then  $\forall \varepsilon > 0$ ,  $\exists x', y' \in L^1 \cap L^\infty$  s.t.  $\|x - x'\|_2 < \varepsilon$ ,  $\|y - y'\|_2 < \varepsilon$  and, as  $|(y', (T_t - I)x')| < \varepsilon$ , for  $0 \leq t < \delta_\varepsilon$ , we get

$$\begin{aligned} |(y, (T_t - I)x)| &\leq |(y - y', (T_t - I)x)| + |(y', (T_t - I)(x - x'))| + |(y', (T_t - I)x')| \\ &\leq \|y - y'\|_2 \|(T_t - I)x\|_2 + |((\hat{T}_t - I)y', x - x')| + |(y', (T_t - I)x')| \\ &\leq \|y - y'\|_2 \|(T_t - I)x\|_2 + \|(\hat{T}_t - I)y'\|_2 \|x - x'\|_2 + |(y', (T_t - I)x')| \\ &\leq \varepsilon(3\|x\|_2 + 3(\|y\|_2 + \varepsilon) + 1). \end{aligned}$$

Hence  $\{T_t\}_{t \geq 0}$  is weakly continuous on  $L^2$  and therefore [D1, proposition 1.23], strongly continuous. Analogously  $\{\hat{T}_t\}_{t \geq 0}$  is strongly continuous on  $L^2$ . They are sub-Markovian semigroups, and the induced extensions of  $T_t|_{L^2 \cap L^\infty}$  and  $\hat{T}_t|_{L^2 \cap L^\infty}$  to  $L^\infty$  are the original semigroups.  $\square$

As we saw in proposition 3.3, if  $M, M^*$  are sub-Markovian operators on  $L^2$ , their extensions to  $L^p$  have norm no greater than 2. If we require a stronger condition of positivity on  $M, M^*$  then their extensions to  $L^p$  are contractive operators, as in the following

**3.8 Theorem.** *If  $M$  and  $M^*$  are sub-Markovian operators on  $L^2$  s.t.*

$$\begin{aligned} (Mx)^*(Mx) &\leq \|M\|_\infty M(x^*x) \\ (M^*x)^*(M^*x) &\leq \|M^*\|_\infty M^*(x^*x), \quad x \in L^2 \cap L^\infty, \end{aligned}$$

then

$$\|M^{(p)}\|_p \leq 1, \quad \|M^{*(p)}\|_p \leq 1, \quad p \in [1, \infty].$$

**Proof.** From [DL1, lemma 3.2] one gets  $\|M|_{L^2 \cap L^\infty}\|_\infty \leq 1$  and  $\|M^*|_{L^2 \cap L^\infty}\|_\infty \leq 1$ . Let  $x \in L^\infty$  and  $\{x_\alpha\} \subset L^2 \cap L^\infty$  s.t.  $x_\alpha \rightarrow x$  weak\* and  $\|x_\alpha\|_\infty \leq \|x\|_\infty$ . Then

$M^{(\infty)}x_\alpha \rightarrow M^{(\infty)}x$  weak\* so that  $\|M^{(\infty)}x\|_\infty \leq \liminf \|M^{(\infty)}x_\alpha\|_\infty \leq \liminf \|x_\alpha\|_\infty \leq \|x\|_\infty$ . In the same way  $\|M^{*(\infty)}\|_\infty \leq 1$ .

Let now  $x \in L^1 \cap L^2$ , as  $B_0 := \{y \in L^2 \cap L^\infty : \|y\|_\infty \leq 1\}$  is weak\* dense in the unit ball of  $L^\infty$ , we have

$$\begin{aligned} \|M^{(1)}x\|_1 &= \sup\{|(y, M^{(1)}x)| : y \in B_0\} \\ &= \sup\{|(M^*y, x)| : y \in B_0\} \\ &\leq \sup\{\|M^*y\|_\infty : y \in B_0\} \|x\|_1 \\ &\leq \|x\|_1 \end{aligned}$$

and, by density and continuity,  $\|M^{(1)}x\|_1 \leq \|x\|_1$ ,  $x \in L^1$ . By interpolation  $\|Mx\|_p \leq \|x\|_p$ ,  $x \in L^2 \cap L^p$  and, by density and continuity,  $\|M^{(p)}x\|_p \leq \|x\|_p$ ,  $x \in L^p$ . An analogous result holds for  $M^{*(p)}$ .  $\square$

**3.9 Remark.** The hypotheses of theorem 3.8 are implied by 2-positivity of the sub-Markovian operators, [DL1].

**3.10 Definition.** Let  $\{\mathcal{E}, \mathcal{D}(\mathcal{E})\}$  be a sesquilinear form on  $L^2(\mathcal{A}, \tau)$ . Then

$$\mathcal{E}^{[n]}([a_{ij}], [b_{ij}]) := \sum_{i,j=1}^n \mathcal{E}(a_{ij}, b_{ij}), \quad a_{ij}, b_{ij} \in \mathcal{D}(\mathcal{E}),$$

is a sesquilinear form on  $L^2(\mathcal{A} \otimes M_n, \tau \otimes tr) \cong L^2(\mathcal{A}, \tau) \otimes L^2(M_n, tr)$ , where  $tr$  is the usual trace on  $n$  by  $n$  matrices.

We say that  $\mathcal{E}$  is  $n$ -Dirichlet if  $\mathcal{E}^{[n]}$  is a Dirichlet form.

**3.11 Lemma.** Let  $\{\mathcal{E}, \mathcal{D}(\mathcal{E})\}$  be a coercive closed form on  $L^2(\mathcal{A}, \tau)$ , and let  $\{G_\alpha\}_{\alpha \geq 0}$  be the associated resolvent.

Then  $\mathcal{E}^{[n]}$  is a coercive closed form and  $\{G_\alpha^{[n]}\}_{\alpha \geq 0}$  is the associated resolvent.

**Proof.** Let us observe that

$$\begin{aligned} \mathcal{E}_\alpha^{[n]}([a_{ij}], G_\alpha^{[n]}[b_{ij}]) &= \mathcal{E}_\alpha^{[n]}([a_{ij}], [G_\alpha b_{ij}]) = \sum_{i,j=1}^n \mathcal{E}_\alpha(a_{ij}, G_\alpha b_{ij}) \\ &= \sum_{i,j=1}^n (a_{ij}, b_{ij}) = ([a_{ij}], [b_{ij}]) \end{aligned}$$

so that  $\{G_\alpha^{[n]}\}_{\alpha \geq 0}$  is the resolvent associated to  $\mathcal{E}^{[n]}$ . Let us now prove that  $\{G_\alpha^{[n]}\}_{\alpha \geq 0}$  is contractive

$$\|\alpha G_\alpha^{[n]}[a_{ij}]\|_2^2 = \|[\alpha G_\alpha a_{ij}]\|_2^2 = \sum_{i,j=1}^n \|\alpha G_\alpha a_{ij}\|_2^2 \leq \sum_{i,j=1}^n \|a_{ij}\|_2^2 = \|[a_{ij}]\|_2^2.$$

Finally let us prove  $\{G_\alpha^{[n]}\}_{\alpha \geq 0}$  satisfies the sector condition [see 1.3]. Let  $[a_{ij}]$ ,  $[b_{ij}] \in L^2(\mathcal{A} \otimes M_n, \tau \otimes tr)_h$ , then

$$\begin{aligned} |([a_{ij}], G_1^{[n]}[b_{ij}])| &= |([a_{ij}], [G_1 b_{ij}])| \\ &= \left| \sum_{i,j=1}^n (a_{ij}, G_1 b_{ij}) \right| \\ &\leq \sum_{i,j=1}^n |(a_{ij}, G_1 b_{ij})| \\ &\leq K \sum_{i,j=1}^n (a_{ij}, G_1 a_{ij})^{1/2} (b_{ij}, G_1 b_{ij})^{1/2} \\ &\leq K \left( \sum_{i,j=1}^n (a_{ij}, G_1 a_{ij}) \right)^{1/2} \left( \sum_{i,j=1}^n (b_{ij}, G_1 b_{ij}) \right)^{1/2} \\ &= K ([a_{ij}], G_1^{[n]}[a_{ij}])^{1/2} ([b_{ij}], G_1^{[n]}[b_{ij}])^{1/2} \end{aligned}$$

that is  $\mathcal{E}^{[n]}$  is a coercive closed form.  $\square$

**3.12 Remark.** Analogous results hold for  $\{G_\alpha^*\}_{\alpha \geq 0}$  and the associated semigroups  $\{T_t\}_{t>0}$  and  $\{T_t^*\}_{t>0}$ , that is  $\{G_\alpha^{[n]}\}_{\alpha \geq 0}$  is the resolvent and  $\{T_t^{[n]}\}_{t>0}$ , and  $\{T_t^{*[n]}\}_{t>0}$ , are the semigroups associated to  $\mathcal{E}^{[n]}$ .

**3.13 Theorem.** Let  $\{\mathcal{E}, \mathcal{D}(\mathcal{E})\}$  be a Dirichlet form and  $\{T_t\}_{t \geq 0}$ ,  $\{T_t^*\}_{t \geq 0}$  the associated semigroups.

Then  $\mathcal{E}$  is  $n$ -Dirichlet  $\iff \{T_t\}$  and  $\{T_t^*\}$  are  $n$ -positive.

**Proof.** From the previous lemma it suffices to show that  $\{T_t\}$  is sub-Markovian and  $n$ -positive  $\iff \{T_t^{[n]}\}$  is sub-Markovian and contractive.

( $\Leftarrow$ ) Let  $x \in L^2$  be such that  $0 \leq x \leq 1$ , then  $0 \leq x \otimes 1 \leq 1 \otimes 1$ , which implies  $0 \leq T_t^{[n]}(x \otimes 1) \leq 1 \otimes 1$  that is  $0 \leq (T_t x) \otimes 1 \leq 1 \otimes 1$  that is  $0 \leq T_t x \leq 1$ .

( $\Rightarrow$ ) First of all, let us observe that  $T_t^{(\infty)[n]}$  is the extension of  $T_t^{[n]}$  to  $L^\infty(\mathcal{A} \otimes M_n, \tau \otimes tr)$  by uniqueness, so that  $T_t^{(\infty)[n]}$  is positive.

Let  $x = [x_{ij}] \in L^2(\mathcal{A} \otimes M_n, \tau \otimes tr)$  be s.t.  $0 \leq x \leq 1$ . Then  $0 \leq T_t^{(\infty)[n]}x \leq T_t^{(\infty)[n]}1$  that is  $0 \leq [T_t x_{ij}] \leq T_t^{(\infty)}1 \otimes 1 \leq 1 \otimes 1$  and the thesis follows.

Finally  $T_t^{[n]} = T_t \otimes 1$  is obviously a contraction on  $L^2(\mathcal{A} \otimes M_n, \tau \otimes tr)$   $\square$

## Section 4. Derivations on square integrable operators.

In this Section we consider derivations on the space  $L^2(\mathcal{A}, \tau)$ .

By this we mean a linear operator

$$\delta : \mathcal{D} \subseteq L^2(\mathcal{A}, \tau) \rightarrow L^2(\mathcal{A}, \tau),$$

where  $\mathcal{D}$  is a subalgebra of  $L^2(\mathcal{A}, \tau) \cap L^\infty(\mathcal{A}, \tau)$ , and  $\delta$  verifies

$$\delta(ab) = a \cdot \delta b + \delta a \cdot b \quad a, b \in \mathcal{D}.$$

We say that a derivation  $\delta$  is closed under the  $\mathcal{C}^1$ , resp. Lipschitz functional calculus if, whenever  $a \in \mathcal{D}_h$ ,  $f(a) \in \mathcal{D}$  for each  $\mathcal{C}^1$ , resp. Lipschitz function  $f$  such that  $f(0) = 0$ .

The domain  $\mathcal{D}$  of a derivation is said *self-adjoint* if it is closed under the  $*$  operation. A dense  $*$ -subalgebra of  $L^2(\mathcal{A}, \tau) \cap L^\infty(\mathcal{A}, \tau)$  is called a *Hilbert algebra*. A derivation  $\delta$  is a  $*$ -*derivation* if  $\mathcal{D}$  is self-adjoint and  $\delta(a^*) = (\delta a)^*$ .

Now we follow an argument in [Sa] which gives rise to a *non-abelian chain rule* (formula 4.3) for the derivation of the functional calculus of a self-adjoint element. Let us fix a self-adjoint element  $a \in \mathcal{A}$  and consider the representation  $\pi_a$  of  $\mathcal{C}_0(\mathbf{R}) \otimes \mathcal{C}_0(\mathbf{R}) \equiv \mathcal{C}_0(\mathbf{R} \times \mathbf{R})$  on  $L^2(\mathcal{A}, \tau)$  given by

$$\pi_a(f \otimes g)b = f(a)bg(a), \quad b \in L^2(\mathcal{A}, \tau).$$

and observe that

$$\text{Range}(\pi_a) \subset \mathcal{A} \vee \mathcal{A}'. \quad (4.1)$$

For each  $f \in \text{Lip}(\mathbf{R})$ , we set

$$\tilde{f}(s, t) = \begin{cases} \frac{f(s) - f(t)}{s - t} & s \neq t \\ f'(t) & s = t \end{cases}. \quad (4.2)$$

We observe that if  $f \in \text{Lip}(\mathbf{R})$  then  $\tilde{f} \in L^\infty(\mathbf{R} \times \mathbf{R})$ , and, if  $f(0) = 0$ , then

$$\|f\|_{\text{Lip}(\mathbf{R})} := \|\tilde{f}\|_\infty \equiv \|f'\|_\infty$$

is a Banach norm on  $\text{Lip}(\mathbf{R})$ . Now we may state the main theorem of this section:

**4.1 Theorem.** *Let  $\delta$  be a closed derivation on  $L^2(\mathcal{A}, \tau)$ ,  $a \in \mathcal{D}_h$ . Then the following properties hold:*

(i)  $\delta$  is closed under the Lipschitz functional calculus.

(ii) For each  $f \in \mathcal{C}_0^1(\mathbf{R})$ ,  $f(0) = 0$ , one has

$$\delta f(a) = \pi_a(\tilde{f})\delta a . \quad (4.3)$$

(iii) For each  $f \in Lip(\mathbf{R})$ ,  $f(0) = 0$ , one has

$$\|\delta f(a)\|_2 \leq \|f\|_{Lip(\mathbf{R})} \|\delta a\|_2 . \quad (4.4)$$

**4.2 Lemma.** Let  $f$  be a Lipschitz continuous function such that  $f(0) = 0$ ,  $\varphi$  a positive  $\mathcal{C}^\infty$  function with support in  $[-1, 1]$  s.t.  $\int \varphi = 1$ . Then, the sequence of mollified functions  $\{f_n\}$ ,  $f_n(t) := f * \varphi_n(t) - f * \varphi_n(0)$ , where  $\varphi_n(t) := n\varphi(nt)$ , verifies the following properties:

- (a)  $f_n(0) = 0$
- (b)  $\|f - f_n\|_\infty \leq \frac{2}{n} \|f\|_{Lip(\mathbf{R})}$
- (c)  $\|f_n\|_{Lip(\mathbf{R})} \leq \|f\|_{Lip(\mathbf{R})}$ .
- (d)  $\tilde{f}_n \rightarrow \tilde{f}$  weak\* in  $L^\infty(\mathbf{R})$

The proof is trivial and is omitted.

**4.3 Lemma.** Let  $\{A, \mathcal{D}(A)\}$  be a closed linear operator on  $\mathcal{H}$ ,  $\{x_n\} \subset \mathcal{D}(A)$  such that  $\|x_n - x\| \rightarrow 0$  and exists  $k > 0$  s.t.  $\|Ax_n\| \leq k$ . Then there exists  $\{w_n\}$  in the convex hull of  $\{x_n\}$  s.t.  $w_n \rightarrow x$  in the graph-norm of  $A$ . As a consequence  $x \in \mathcal{D}(A)$  and  $\|Ax\| \leq k$ .

**Proof.** Let us denote by  $\|\cdot\|_A$  the graph-norm,  $\|y\|_A := \|y\| + \|Ay\|$ ,  $y \in \mathcal{D}(A)$ . By the hypotheses, the sequence  $x_n$  is bounded in the graph norm, therefore the set of limit points in the weak topology of  $\{\mathcal{D}(A), \|\cdot\|_A\}$  is not empty. By the Banach-Saks theorem (cf. [DS], Theorem. V.3.14), for any such limit point  $y$ , there exists a sequence  $w_n$  in the convex hull of  $\{x_n\}$  such that  $w_n \rightarrow y$  in the graph-norm, hence in the Hilbert norm, therefore  $y = x$ , that is  $x \in \mathcal{D}(A)$ . Besides

$$\|Ax\| = \lim \|Aw_n\| \leq k$$

and the thesis follows. □

**Proof of Theorem 4.1.** First we observe that, since  $a \in \mathcal{A}$ , we may replace  $\mathcal{C}_0^1(\mathbf{R})$  by  $\mathcal{C}^1(I)$ ,  $I := [-\|a\|, \|a\|]$ . Then, equation (4.3) makes sense also for polynomials, and

we check it for  $f(t) := t^n$ :

$$\begin{aligned}
\delta(a^n) &= \sum_{j=0}^{n-1} a^j (\delta a) a^{n-j-1} = \\
&= \sum_{j=0}^{n-1} \pi_a(s^j t^{n-j-1}) \delta a \\
&= \pi_a \left( \frac{s^n - t^n}{s - t} \right) \delta a = \pi_a(\tilde{f}) \delta a.
\end{aligned}$$

By linearity, (4.3) holds for all polynomials  $p$  such that  $p(0) = 0$ . Finally we observe that, for all such polynomials,

$$\begin{aligned}
\|\pi_a(\tilde{p}) \delta a\|_2 &\leq \|\pi_a(\tilde{p})\| \|\delta a\|_2 \\
&= \|p\|_{Lip(I)} \|\delta a\|_2 \\
&= \|p\|_{C^1(I)} \|\delta a\|_2.
\end{aligned}$$

Therefore, if  $p_n$  is a sequence of polynomials converging to a  $\mathcal{C}^1$  function  $f$  in the  $\mathcal{C}^1(I)$  norm, then  $\delta(p_n(a))$  is a Cauchy sequence w.r.t. the graph norm of  $\delta$ , and (ii) follows by continuity.

In particular, we proved that  $\delta$  is closed under the  $\mathcal{C}^1$  functional calculus. Therefore, if  $f, \varphi_n$  are as in lemma 4.2, formula (4.3) applies to  $f_n$ , and we get, using lemma 4.2c,

$$\|\delta f_n(a)\|_2 = \|\pi_a(\tilde{f}_n) \delta a\|_2 \leq \|f\|_{Lip(\mathbf{R})} \|\delta a\|_2.$$

Now we prove that

$$\|f(a) - f_n(a)\|_2 \rightarrow 0. \quad (4.5)$$

Indeed, choosing  $\|f - f_n\|_\infty \leq \varepsilon^3$ , we have

$$\begin{aligned}
\|f(a) - f_n(a)\|_2^2 &= \tau \left( \int |f - f_n|^2(\lambda) de(\lambda) \right) \leq \\
&\leq \int_{-\varepsilon}^{\varepsilon} \left( \left| \frac{f(\lambda)}{\lambda} \right| + \left| \frac{f_n(\lambda)}{\lambda} \right| \right)^2 d\mu(\lambda) + \frac{1}{\varepsilon^2} \int_{|\lambda| \geq \varepsilon} |f - f_n|^2 d\mu \leq \\
&\leq 4 \|f\|_{Lip(\mathbf{R})}^2 \mu([- \varepsilon, \varepsilon]) + \|a\|_2^2 \varepsilon,
\end{aligned}$$

where the measure  $\mu$  is defined by  $\mu(\Omega) := \tau(\int_\Omega \lambda^2 de(\lambda))$ . Since  $\mu(\{0\}) = 0$  and  $\mu(\mathbf{R}) = \|a\|_2$  by definition, we get  $\mu([- \varepsilon, \varepsilon]) \rightarrow 0$ , which proves (4.5).

Finally we apply lemma 4.3 to the sequence  $f_n(a)$  in the domain of  $\delta$ , and (i) and (iii) are proven.  $\square$

The following corollary is a consequence of theorem 4.1.

**4.4 Corollary.** *Let  $\delta_n$ ,  $n = 1, \dots, N$ , be closed derivations. If  $\delta := \sum_n \delta_n$ ,  $\mathcal{D}(\delta) = \cap_n \mathcal{D}(\delta_n)$ , then  $\delta$  is closed under the Lipschitz functional calculus and, for each  $f \in Lip(\mathbf{R})$ ,  $f(0) = 0$ ,*

$$\|\delta f(a)\|_2 \leq \|f\|_{Lip(\mathbf{R})} \|\delta a\|_2 \quad \forall a \in \mathcal{D}(\delta)_h$$

**Proof.** By hypothesis  $\delta$  is closed under Lipschitz functional calculus and (4.3) holds for  $\mathcal{C}^1$  functions. Let  $f$  and  $f_n$  be as in lemma 4.2. Then, as in the proof of theorem 4.1, we get

$$\|\delta f_n(a)\|_2 = \|\pi_a(\tilde{f}_n)\delta a\|_2 \leq \|f\|_{Lip(\mathbf{R})} \|\delta a\|_2$$

and  $\|f_n(a) - f(a)\|_2 \rightarrow 0$ . Then, by lemma 4.3, we get a sequence  $\{h_n\}$  in the convex hull of  $\{f_n\}$  such that  $h_n(a) \rightarrow f(a)$  in the graph-norm of  $\delta_1$ . Since  $\{h_n(a)\}$  is bounded in the graph-norm of  $\delta_2$ , applying again lemma 4.3, we find a sequence in the convex hull of  $\{f_n\}$  converging to  $f(a)$  in the graph-norms of  $\delta_1$  and  $\delta_2$ . Iterating this procedure  $N$  times, we find a sequence  $\{g_n\}$  in the convex hull of  $\{f_n\}$  s.t.  $g_n(a)$  converges to  $f(a)$  in the graph-norms of  $\delta_i$ , for all  $i$ . Therefore  $g_n(a) \rightarrow f(a)$  in the graph-norm of  $\delta$ . Finally

$$\|\delta f(a)\|_2 = \lim \|\delta g_n(a)\|_2 \leq \|f\|_{Lip(\mathbf{R})} \|\delta a\|_2.$$

□

**4.5 Remark.** We would like to compare theorem 4.1 with an analogous result of Powers ([Po], cf. theorem 1.6.2 in [B]) for the derivations on a  $C^*$ -algebra. While the difference in the formulation of the non-abelian chain-rule (equation 4.3) is just a matter of taste, the difference on the allowed functional calculus depends on the different norms. Indeed, let  $\mathcal{M}$  be the  $C^*$ -algebra generated by a self-adjoint element in  $\mathcal{A}$ . The representation of the tensor product  $\mathcal{M} \otimes \mathcal{M}$  given by the left and right actions  $\mathcal{M}$  on  $L^2(\mathcal{A}, \tau)$  extends to a representation of the  $C^*$ -tensor product. This guarantees that equation (4.3) holds for the closure of the polynomials in the appropriate norm, *i.e.* for  $\mathcal{C}^1$  functions. On the contrary, if the abelian  $C^*$ -algebra  $\mathcal{M}$  acts on  $\mathcal{A}$ , the tensor product is embedded in the Banach algebra  $\mathcal{B}(\mathcal{A})$ , and this embedding is not necessarily continuous in the  $C^*$  norm. As a consequence, formula (4.3) does not necessarily hold for  $\mathcal{C}^1$  functions, as it is shown by McIntosh [Mc].

Theorem 4.1 gives a general answer to the problem of the Lipschitz functional calculus of a self-adjoint operator in the domain of a derivation. On the other hand there is a trivial form of “Lipschitz” functional calculus that makes sense also for non

self-adjoint operators, *i.e.* the modulus of an operator. Now we are going to study this question.

**4.6 Lemma.** *Let  $\delta$  be a derivation on  $\mathcal{D}$ . Then the operator  $\delta^\dagger : \mathcal{D}^* \rightarrow L^2(\mathcal{A}, \tau)$  defined by*

$$\delta^\dagger a := (\delta a^*)^* , \quad a \in \mathcal{D}^*$$

*is a derivation. If  $\delta$  is closed (closable), also  $\delta^\dagger$  is.*

**Proof.** The Leibnitz rule for  $\delta^\dagger$  follows by a straightforward calculation. The equivalence of the closability properties follows by the equality

$$\|a\|_{\delta^\dagger} = \|a^*\|_\delta , \quad \forall a \in \mathcal{D}^*$$

□

**4.7 Lemma.** *Let  $\delta$  be a closed derivation on a self-adjoint domain  $\mathcal{D}$ . Then*

$$\begin{pmatrix} \delta^\dagger & 0 \\ 0 & \delta \end{pmatrix} : \mathcal{D} \otimes M_2 \rightarrow L^2(\mathcal{A}, \tau) \otimes M_2$$

*is a closed derivation.*

**Proof.** Follows immediately by the property

$$\left\| \begin{pmatrix} \delta^\dagger & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|^2 = \|\delta a^*\|^2 + \|\delta b^*\|^2 + \|\delta c\|^2 + \|\delta d\|^2.$$

□

**4.8 Theorem.** *Let  $\delta$  be a closed derivation on a self-adjoint domain  $\mathcal{D}$ . Then, if  $a \in \mathcal{D}$ ,  $|a| \in \mathcal{D}$  and*

$$\|\delta|a|\|_2 \leq \sqrt{2}\|\delta a\|_2. \quad (4.6)$$

**Proof.** Consider the functions

$$\begin{aligned} \varphi_n(t) &= \sqrt{t + \frac{1}{n^2}} - \frac{1}{n} & t \geq 0 \\ \psi_n(t) &= \sqrt{t^2 + \frac{1}{n^2}} - \frac{1}{n} & t \in \mathbf{R} . \end{aligned}$$



Since  $a \in \mathcal{D}$ ,  $a^* \in \mathcal{D}$ , and therefore  $a^*a \in \mathcal{D}$  and

$$\psi_n(|a|) = \varphi_n(a^*a) \in \mathcal{D}$$

by the theorem on the Lipschitz functional calculus.

Now consider the following chain of inequalities:

$$\begin{aligned} \|\delta\psi_n(|a|)\|_2^2 &\leq \|\delta\psi_n(|a|)\|_2^2 + \|\delta\psi_n(|a^*|)\|_2^2 = \\ &= \|\delta^\dagger\psi_n(|a|)\|_2^2 + \|\delta\psi_n(|a^*|)\|_2^2 = \\ &= \left\| \begin{pmatrix} \delta^\dagger & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} \psi_n(|a|) & 0 \\ 0 & \psi_n(|a^*|) \end{pmatrix} \right\|_2^2 = \\ &= \left\| \begin{pmatrix} \delta^\dagger & 0 \\ 0 & \delta \end{pmatrix} \psi_n \left( \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} \right) \right\|_2^2 \leq \\ &= \left\| \begin{pmatrix} \delta^\dagger & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} \right\|_2^2 = \\ &= \|\delta a\|_2^2 + \|\delta^\dagger a^*\|_2^2 = \\ &= 2\|\delta a\|_2^2 \end{aligned}$$

where the main inequality follows by the theorem on Lipschitz functional calculus applied to the derivation  $\begin{pmatrix} \delta^\dagger & 0 \\ 0 & \delta \end{pmatrix}$  mentioned in lemma 4.7.

Now it is easy to see that  $\psi_n(|a|) \rightarrow |a|$  in the  $L^2$  norm, and therefore the result follows from lemma 4.3.  $\square$

We remark that, according to the terminology in [DL1], theorems 4.1 and 4.8 may be rephrased as follows: each closed derivation on a Hilbert algebra is a Dirichlet derivation.

A natural question related to theorem 4.1 is the following: when the non-abelian chain rule given in (4.3) extends to the Lipschitz functional calculus? The first problem is that  $\tilde{f}$  is not necessarily in the domain of  $\pi$ , when  $f$  is a Lipschitz function. Indeed  $\pi$  may easily be extended to the  $C^*$  tensor product of  $L^\infty(\mathbf{R})$  with itself, but this space is smaller than  $L^\infty(\mathbf{R} \times \mathbf{R})$  and therefore does not contain  $\tilde{f}$  in general. Even though we do not try to give a general answer to the previous question, in the following proposition we mention two extremal cases in which the addressed question has a positive answer, the abelian case and the type I factor case.

**4.9 Proposition.** *Let  $\mathcal{A}$  be either an abelian algebra or a type I factor, and  $\delta$  a closed derivation on  $L^2(\mathcal{A}, \tau)$ . Then, for each self-adjoint  $a$  in the domain of  $\delta$ , the*

map  $\pi_a$  extends to  $L^\infty(\mathbf{R} \times \mathbf{R})$ . Therefore, the non abelian chain-rule given in (4.3) extends to Lipschitz functional calculus.

**Proof.** It is well known that, if  $\mathcal{A}$  is a type I factor, the map  $\pi_a$  extends to a normal representation of  $L^\infty(\sigma(a) \times \sigma(a))$ , for each self-adjoint  $a$  in  $\mathcal{A}$ . Then, let  $f$  and  $f_n$  be as in lemma 4.2. By lemma 4.2d,  $\pi_a(\tilde{f}_n)\delta a \rightarrow \pi_a(\tilde{f})\delta a$  weakly in  $L^2(\mathcal{A}, \tau)$ , and also  $\delta f_{n_k}(a) \rightarrow \delta f(a)$  weakly in  $L^2(\mathcal{A}, \tau)$  for a suitable subsequence, as it is shown in lemma 4.3. Then the thesis holds. If  $\mathcal{A}$  is abelian, formula 4.3 becomes the usual chain rule, and by normality of the map  $f \rightarrow f(a)$ ,  $a \in \mathcal{A}_h$ , we get the thesis.  $\square$

**4.10 Remark.** We remark that the key property we used in the proof of the factor case, i.e. the fact that the von Neumann algebra generated by the right and left action of any abelian subalgebra of  $\mathcal{A}$  on the standard representation of  $\mathcal{A}$  is isomorphic to the tensor product of  $\mathcal{A}$  with itself, is a characterization of the type I, therefore the property we are studying is probably confined to type I algebras.

We conclude this section with an example of a simple Dirichlet form associated with a general derivation.

**4.11 Proposition.** *Let  $\delta$  be a closed derivation on a Hilbert algebra  $\mathcal{D}$ . Then the form*

$$\mathcal{E}(x, y) = \operatorname{Re}(\delta x, \delta y) + \operatorname{Im}(\delta x, \delta y) \quad (4.7)$$

*is a Dirichlet form.*

**Proof.** The form  $\mathcal{E}$  is closed *iff* its symmetric part  $\operatorname{Re}(\delta x, \delta y)$  is, that is *iff*  $\delta$  is closed. The weak sector condition follows by

$$|\mathcal{E}(x, y)|^2 \leq 2|(\delta x, \delta y)|^2 \leq 2\mathcal{E}(x, x)\mathcal{E}(y, y).$$

Now let us define the operators

$$\begin{aligned} d_1 a &= \frac{\delta a + \delta^\dagger a}{2} \\ d_2 a &= \frac{\delta a - \delta^\dagger a}{2i} \end{aligned} \quad a \in \mathcal{D}.$$

It is clear that  $d_1, d_2$  are  $*$ -derivations and

$$\mathcal{E}(x, y) = \sum_{i,j \in \{1,2\}} a_{ij}(d_i x, d_j y),$$

where  $(a_{ij}) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ . Now Dirichlet property follows from the general theorem in the following section.  $\square$

As the proof of the preceding proposition shows, the study of the Dirichlet form in (4.7) may be reduced to the case of  $*$ -derivations. Therefore in the following section only  $*$ -derivations will be considered.

## Section 5. Explicit constructions of Dirichlet forms.

The aim of this section is to describe a class of Dirichlet forms which can be considered as the non commutative generalization of a class of (generally non symmetric) commutative Dirichlet forms studied in [MR]. At the same time these non commutative examples also extend previous ones constructed in [DL1].

We start considering the following leading example taken from the commutative context. Let  $A = [a_{ij}]$  be an element of  $L^1_{loc}(U) \otimes M_n$ ,  $U \subset \mathbf{R}^n$  open. Then we define the bilinear form on  $\mathcal{C}_o^\infty(U)$

$$\mathcal{E}(u, v) := \sum_{i,j=1}^n \int a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx . \quad (5.1)$$

Under the following simple assumptions: there exists  $0 < \nu < \infty$  such that

$$\begin{cases} \sum_{i,j=1}^n \tilde{a}_{ij}(x) \xi_i \xi_j \geq \nu \|\xi\|^2 & \forall \xi = (\xi_1, \dots, \xi_n) \\ \tilde{a}_{ij} \in L^\infty(U, dx) & 1 \leq i, j \leq n. \end{cases} \quad (5.2)$$

where  $\tilde{A}$  (resp.  $\check{A}$ ) is the symmetric (resp. antisymmetric) part of  $A$ , it can be proven that the form  $\{\mathcal{E}, D(\mathcal{E})\}$  is closable and its closure is a Dirichlet form (cf.[MR]).

Now we discuss some generalizations of the preceding example to the non commutative context. The general setting is the following: we have a sesquilinear form of the type

$$\mathcal{E}(x, y) := \sum_{i=1}^n (d_i x, a_{ij} d_j y) \quad (5.1')$$

where  $d_i$ ,  $i = 1 \dots n$ , is a family of  $*$ -derivations on  $L^2(\mathcal{A}, \tau)$  and the  $a_{ij}$ 's belong to the center of  $\mathcal{A}$ . In this case condition (5.2) is replaced by its proper non-commutative analogue:

$$\begin{cases} \tilde{A} \geq \nu I , \\ \tilde{a}_{ij} \in L^\infty(\mathcal{A}, \tau) , \quad 1 \leq i, j \leq n. \end{cases} \quad (5.2')$$

Our first result is theorem 5.1. This theorem deals with *general*  $*$ -derivations with a dense common domain. In this case, because of such generality, we need stronger requirements in order to get closedness for the form (5.1'). This is obtained by asking the symmetric part of the matrix  $A$  to be the identity matrix, which makes the first condition in (5.2') automatically fulfilled.

In theorem 5.2 the derivations  $d_i$  are given by commutators  $[z_i, \cdot]$ , where the  $z_i$ 's are skew-symmetric elements in  $L^2 + L^\infty$ , which provides the closedness of the form  $\mathcal{E}$ . In this case conditions (5.2') suffice to get the result. Such a theorem is a non symmetric extension of theorem 6.10 in [DL1].

**5.1 Theorem.** *Assume we are given a family*

$$d_i : \mathcal{D}_i \subset L^2(\mathcal{A}, \tau) \rightarrow L^2(\mathcal{A}, \tau) \quad i = 1, \dots, n$$

*of  $*$ -derivations over Hilbert algebras  $\mathcal{D}_i$  such that*

*(a) each  $d_i$  is closable*

*(b)  $\mathcal{D} := \cap_{i=1}^n \mathcal{D}_i$  is dense*

*and consider the form  $\mathcal{E}$  given by*

$$\begin{cases} D(\mathcal{E}) := \mathcal{D} \\ \mathcal{E}(x, y) := \sum_{i=1}^n (d_i x, d_i y) + \sum_{i,j=1}^n (d_i x, c_{ij} d_j y) \end{cases} \quad (5.3)$$

*where the  $c_{ij}$ 's are self-adjoint elements in the center of  $\mathcal{A}$  such that  $c_{ij} = -c_{ji}$ . Then the form is closable and its closure is a Dirichlet form.*

**Proof.** Sesquilinearity of  $\mathcal{E}$  being evident, we prove real positivity. We have

$$\mathcal{E}(x, x) = \sum_{i=1}^n \|d_i x\|^2 + \tau \otimes \text{tr}(CB(x, x)) \quad (5.4)$$

with  $C = [c_{ij}] \in L^\infty(\mathcal{A}, \tau) \otimes M_n$  and  $B(x, y) \in L^1(\mathcal{A}, \tau) \otimes M_n$  is given by  $B(x, y) = [d_i y d_j x]$ . Since  $C$  is a real antisymmetric matrix and  $B(x, x)$  is a real symmetric matrix when  $x \in \mathcal{D}_h$ , the last term in the right hand side of (5.4) vanishes, and the positivity of  $\mathcal{E}$  follows. Because of hypothesis (a) the form  $\tilde{\mathcal{E}}$  is closable (see e.g. [DL1]) and therefore, by definition,  $\mathcal{E}$  is closable. We now prove that the weak sector condition holds for  $\mathcal{E}$ , i.e. there exists  $0 < K < \infty$  such that

$$|\tilde{\mathcal{E}}(x, y)| \leq K \tilde{\mathcal{E}}_1(x, x)^{1/2} \tilde{\mathcal{E}}_1(y, y)^{1/2}$$

for all  $x, y \in \mathcal{D}_h$ , where  $\tilde{\mathcal{E}}_1(x, y) := \tilde{\mathcal{E}}(x, y) + (x, y)$ .

Setting  $M := \|C\|_\infty$  and applying Hölder and Schwartz inequalities we get

$$\begin{aligned} |\check{\mathcal{E}}(x, y)| &= |\tau \otimes \text{tr}(CB(x, y))| \leq M \|B(x, y)\|_1 \\ &\leq M \sum_{i,j=1}^n \|B(x, y)_{ij}\|_1 \leq M \sum_{i,j=1}^n \|d_j x\|_2 \|d_i y\|_2 \\ &\leq nM \left( \sum_{j=1}^n \|d_j x\|_2^2 \right)^{1/2} \left( \sum_{i=1}^n \|d_i y\|_2^2 \right)^{1/2} \leq nM \tilde{\mathcal{E}}_1(x, x)^{1/2} \tilde{\mathcal{E}}_1(y, y)^{1/2}. \end{aligned}$$

It remains to prove that the closure of the form (5.3) is a Dirichlet form. Let  $\bar{\mathcal{E}}$  denote the closure of the form  $\mathcal{E}$ , which is obtained replacing the  $d_i$ 's with their closures  $\bar{d}_i$  (cf. [DL1]). We notice that if  $x \in \mathcal{D}(\bar{\mathcal{E}})$  then  $\varphi_0(x) := x^+ \wedge 1 \in \mathcal{D}(\bar{\mathcal{E}})$  because this holds for each  $\mathcal{D}_i$ . Hence we claim that

$$\bar{\mathcal{E}}(x \mp \varphi_0(x), x \pm \varphi_0(x)) \geq 0 \quad \forall x \in \mathcal{D}(\bar{\mathcal{E}})_h. \quad (5.5)$$

First we observe that the matrix  $[(\bar{d}_i x, c_{ij} \bar{d}_j \varphi_0(x))]$  is antisymmetric. It is enough to show this replacing  $\varphi_0$  with a smooth approximation  $\varphi$ . Then, by equations (4.1) and (4.3) and by the hypotheses on  $C$ , we get

$$\begin{aligned} (\bar{d}_i x, c_{ij} \bar{d}_j \varphi(x)) &= (\bar{d}_i x, c_{ij} \pi(\tilde{\varphi}) \bar{d}_j x) \\ &= (\pi(\tilde{\varphi}) \bar{d}_i x, c_{ij} \bar{d}_j x) \\ &= (\bar{d}_i \varphi(x), c_{ij} \bar{d}_j x) \\ &= -(\bar{d}_j x, c_{ji} \bar{d}_i \varphi(x)). \end{aligned}$$

Therefore, using again the antisymmetry of  $C$ , we have

$$\bar{\mathcal{E}}(x \mp \varphi_0(x), x \pm \varphi_0(x)) = \sum_{i=1}^n (\bar{d}_i(x \mp \varphi_0(x)), \bar{d}_i(x \pm \varphi_0(x)))$$

If we set  $B_{ij}^\mp := (\bar{d}_i(x \mp \varphi_0(x)), \bar{d}_j(x \pm \varphi_0(x)))$ , the left hand side in (5.5) is just the trace of  $B^\mp$ , therefore (5.5) follows if we prove that  $B^\mp$  is a positive definite real-valued matrix. Indeed, for any  $(t_1, \dots, t_n) \in \mathbf{R}^n$  and setting  $d := \sum_1^n \bar{d}_i$ , we have

$$\begin{aligned} \sum_{i,j=1}^n B_{ij}^\mp t_i t_j &= ((\sum_{i=1}^n t_i \bar{d}_i)(x \mp \varphi_0(x)), (\sum_{j=1}^n t_j \bar{d}_j)(x \pm \varphi_0(x))) \\ &= (d(x \mp \varphi_0(x)), d(x \pm \varphi_0(x))) \\ &= (dx, dx) - (d\varphi_0(x), d\varphi_0(x)) \geq 0 \end{aligned}$$

by corollary 4.4, and this ends the proof.  $\square$

**5.2 Theorem.** *Let  $z_1, \dots, z_n$  be skew-adjoint elements in  $L^2 + L^\infty$ , define*

$$d_i(x) := z_i x - x z_i \quad \forall x \in L^2 \cap L^\infty$$

*and let  $A = [a_{ij}]$  be a matrix of self-adjoint elements in the center of  $\mathcal{A}$  such that condition (5.2') holds. Then, the form*

$$\left\{ \begin{array}{l} \mathcal{D}(\mathcal{E}) := L^2 \cap L^\infty \\ \mathcal{E}(x, y) := \sum_{i,j=1}^n (d_i x, a_{ij} d_j y) \end{array} \right.$$

*is closable and its closure is a Dirichlet form.*

**Proof.** Let us denote by  $B$  the square root of the symmetric part of  $A$  as an element in  $L^\infty(\mathcal{A}, \tau) \otimes M_n$ . We also set  $\delta_i := \sum b_{ij} d_j$  and  $C := B^{-1} \tilde{A} B^{-1}$ . We notice that, since  $\tilde{A}$  is coercive,  $B^{-1}$  is bounded and  $C$  is bounded, real and skew-symmetric. Then,

$$\mathcal{E}(x, y) = \sum_{i=1}^n (\delta_i x, \delta_i y) + (\delta_i x, c_{ij} \delta_j y).$$

Since  $z_i \in L^2 + L^\infty$ ,  $i = 1, \dots, n$  and  $B$  is bounded,  $w_i := \sum b_{ij} z_j \in L^2 + L^\infty$ . Then  $\delta_i$ , being implemented by  $w_i$ , is a closable derivation on  $L^2 \cap L^\infty$  (see e.g. [DL1]) and the thesis follows by theorem 5.1.  $\square$

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